where $K_1 = 0.0225$ at $(di - dp)/dc \leqq 1/24$, and 0.012 for $(di - dp)/dc > 1/24$. It therefore appears that flooding rates in a rotating disk column may be estimated from holdup data, and vice versa.

NOTATION

- a = interfacial area of contact between two fluid phases
- dc = inside diameter of the column
- di = diameter of stator opening
- dp = diameter of rotor disk
- g = acceleration due to gravity
- $(H.T.U.)_o$ = over-all height of a transfer unit
- h = compartment height
- K = over-all mass transfer coefficient
- K_1 = coefficient in characteristic velocity correlation
- k_1 = coefficient in holdup correlation
- m = constant
- N = speed of disk rotation
- V = superficial liquid velocty, cu. ft./(hr.) (sq.ft. of column cross section)
- $\overline{V}, \overline{V}_N$ = characteristic velocity (average) for a given system in a given column geometry and rotor speed, estimated from Equations (1) and (7) respectively, ft./hr.
- $\overline{V}_N'$ = characteristic velocity estimated from Equation (1) for an individual run with a given system in a given column geometry and rotor speed, ft./hr.
- X = fractional holdup of the dispersed phase in that portion of the column volume which may be occupied by liquid
- μ = viscosity
- γ = interfacial tension
- ρ = density
- $\Delta\rho$ = density difference between the dispersed and the continuous phase

Subscripts

- c = continuous phase
- d = discontinuous phase

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Manuscript received April 7, 1959; revision received November 21, 1960; paper accepted November 23, 1960. Paper presented at A.I.Ch.E. Atlantic City meeting.

Flow of Non-Newtonian Fluids in a Magnetic Field

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The analytical solution to the equation of motion is given for the steady laminar flow of a uniformly conducting incompressible non-Newtonian fluid between two parallel planes. The fluid is under the influence of a constant pressure gradient and is subjected to a steady magnetic field perpendicular to the direction of motion. Two non-Newtonian models are considered: the Bingham plastic model and the power-law model. Flow rates and the velocity profiles for various values of the Hartmann number and the generalized Hartmann number are presented and compared with those corresponding to Newtonian fluids.

Since the time of convenient creation of an ideal fluid to the present development of the applied science known as *rheology*, the restrictions imposed on the number of variables affecting the distribution of shear stress in a given flow have been relaxed systematically. In the past years several simple flow problems of classical hydrodynamics have received new attention in the more general context of magnetohydrodynamics. It is sufficient to mention in this connection the work of Hartmann and Lazarus (*1*) on the flow between two parallel walls, its extension to the case of flow in a straight pipe by Shercliff (*2, 3*), and the work of Bleviss (*4*) on the Couette flow. Evidently the inevitable extension of the study of the Newtonian fluid flow in the presence of a magnetic field to the flow of non-Newtonian fluids further relaxes the restrictions imposed on the shear stress and on the other characteristics of flow. It should be noted however that exact solutions for problems of this nature exist for special cases only. These cases may not correspond to a specific case of practical interest. The principal thought behind these investigations is that the significant physical features of the flow will stand out clearly, unobscured by a mathematical thicket, and that the results so obtained will serve as a good approximation to the physical model of interest in predicting the qualitative fluid behavior, and will guide one in facing more complicated situations.

The study of the motion of non-Newtonian fluids in the absence as well as in the presence of a magnetic field has applications in many areas. A few examples are the flow of nuclear fuel slurries, flow of liquid metals and alloys such as the flow of gallium at ordinary temperatures (30°C.), flow of plasma, flow of mercury amalgams,

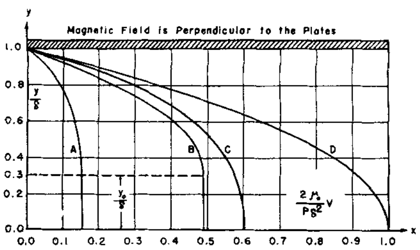

Fig. 1. Comparison of velocity profiles for Bingham and Newtonian fluids at the same pressure gradient. (A) for $N_{Ha} = 3$, $y_o/\delta = 0.3$; (B) for $N_{Ha} = 0$, $y_o/\delta = 0.3$; (C) for $N_{Ha} = 3$, $y_o/\delta = 0.0$; (D) for $N_{Ha} = 0$, $y_o/\delta = 0.0$.

handling of biological fluids, flow of blood—a Bingham fluid with some thixotropic behavior, coating of paper, plastic extrusion, and lubrication with heavy oils and greases.

Another important field of application is the electromagnetic propulsion. Basically an electromagnetic propulsion system consists of a power source, such as a nuclear reactor, a plasma, and a tube through which the plasma is accelerated by electromagnetic forces. The study of such systems, which is closely associated with magnetochemistry, requires a complete understanding of the equation of state and transport properties such as diffusion, shear stress-shear rate relationship, thermal conductivity, electrical conductivity, and radiation. Some of these properties will undoubtedly be influenced by the presence of an external magnetic field which sets the plasma in hydrodynamic motion.

The formulation of shear stress for non-Newtonian fluids is a difficult problem which has not progressed very far from a theoretical standpoint. Nevertheless for want of a more fundamental understanding several empirical descriptions have become well established rheological models. The equations which have found widespread use are (5)

Bingham plastic

$$\tau = \overline{+}\, \tau_o - \mu_o \frac{dv}{dy} \text{ for } |\tau| > \tau_o \quad (1)$$

Ostwald-de Waele power-law model

$$\tau = -m \left| \frac{dv}{dy} \right|^{n-1} \frac{dv}{dy} \quad (2)$$

Eyring model

$$\tau = \frac{1}{m'} \text{arc sinh} \left[-\frac{1}{n'} \frac{dv}{dy} \right] \quad (3)$$

Each of these equations contain two rheological constants which have to be determined experimentally.

As far as the author is aware, problems of non-Newtonian fluid flow in the more general context of magnetohydrodynamics have remained untouched. The present paper studies the laminar flow of Bingham plastics and pseudoplastic and dilatant substances between two parallel planes under the influence of a constant pressure gradient and a steady magnetic field perpendicular to the direction of motion. Throughout this study the generalized magnetic Reynolds number is assumed to be sufficiently small.

EQUATIONS OF MOTION

The basic equations governing the motion of an electrically conducting incompressible non-Newtonian fluid in the presence of a magnetic field are

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)\, v = -\frac{1}{\rho} \nabla p + \frac{\kappa}{\rho} E + \frac{\lambda}{\rho} JxH - \nabla \tau + F \quad (4)$$

$$\nabla \cdot v = 0 \quad (5)$$

$$\nabla x H = J + \frac{\partial \epsilon E}{\partial t} \quad (6)$$

$$\nabla x E = -\frac{\partial \lambda H}{\partial t} \quad (7)$$

$$J = \sigma\, (E + \lambda\, vxH) + \kappa v \quad (8)$$

$$\nabla \cdot J + \frac{\partial \kappa}{\partial t} = 0 \quad (9)$$

In the case of a steady fluid flow above equations may be simplified to

$$(v \cdot \nabla) v = -\nabla \left(\frac{p}{\rho} + \Omega \right) + \frac{\lambda}{\rho} JxH - \nabla \cdot \tau \quad (10)$$

$$\nabla \cdot v = 0 \quad (11)$$

$$\nabla x H = J \quad (12)$$

$$\nabla x E = -\frac{\partial \lambda H}{\partial t} \quad (13)$$

$$J = \sigma\, (E + \lambda\, vxH) \quad (14)$$

$$\nabla \cdot J = 0 \quad (15)$$

in a cartesian coordinate system (x, y, z), which is at rest with respect to the fluid at infinity. It is evident that there are only thirteen independent equations for the thirteen quantities v, E, H, J, and p. If the only other body force is gravity and if there is no free surface, body forces can be cancelled in the motion of a homogeneous fluid provided the pressure is taken to mean the difference between that in motion and that at rest. If one takes the parallel planes as $y = \pm\, \delta$, supposes that the velocity v is in the x-direction, and that a uniform magnetic field H_o is imposed perpendicular to the bounding plates, then the field acquires a component H_x, parallel to the motion, and a uniform electric field E in the z-direction. Hence the equation of motion for the case under consideration simply reduces to

$$-\frac{\partial p}{\partial x} - \frac{\partial \tau}{\partial y} + \lambda\, \sigma\, H_o\, (E - \lambda\, v\, H_o) = 0 \quad (16)$$

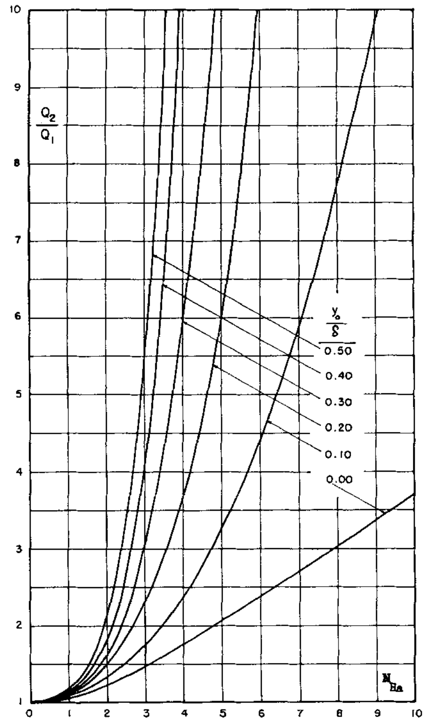

Fig. 2. The ratio Q_2/Q_1 as a function of N_{Ha} and y_o/δ for the Bingham plastics.

As in most electromagnetic problems involving conductors, other than those concerned with the rapid oscillations, Maxwell's displacement currents are neglected in the equation of continuity of charge, so that electric currents are regarded as flowing in closed conduits.

FLOW OF BINGHAM PLASTICS

For the Bingham plastic substitution of Equation (1) into Equation (16) and subsequent integration using the fact that $v = 0$ at $y = \delta$, and $dv/dy = 0$ at $y \leqq y_o$, that is for $|\tau| \leqq \tau_o$, gives

$$v = \frac{(P + \lambda\, \sigma\, E\, H_o)\, \delta^2}{\mu_o\, N^2_{Ha}} \left[1 - \frac{\cosh \left[\frac{y}{y_o} - 1 \right] \frac{y_o}{\delta} N_{Ha}}{\cosh \left(1 - \frac{y_o}{\delta} \right) N_{Ha}} \right] \text{ for } y \geqq y_o \quad (17)$$

which is symmetrical with respect to x-axis.

From Equations (1) and (17) one obtains the shear stress distribution

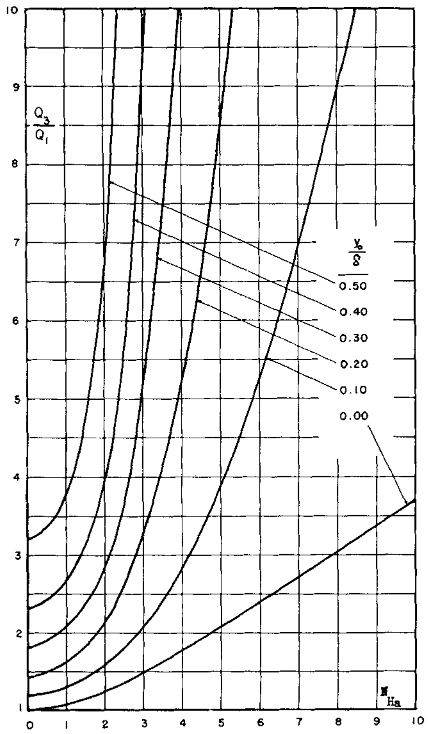

Fig. 3. The ratio Q_B/Q_1 as a function of N_{Ha} and y_o/δ for the Bingham plastics.

$$\tau = \tau_o + \frac{(P + \lambda \sigma E H_o)\delta}{N_{Ha}} \cdot \frac{\sinh\left(\frac{y}{y_o} - 1\right)\frac{y_o}{\delta} N_{Ha}}{\cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha}} \quad (18)$$

in which τ_o and y_o are related by the expression ($\tau = 0$ at $y = 0$)

$$\tau_o = \frac{(P + \lambda \sigma E H_o)\delta}{N_{Ha}} \cdot \frac{\sinh \frac{y_o}{\delta} N_{Ha}}{\cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha}} \quad (19)$$

For the values of y between zero and y_o velocity is given by the following expression:

$$v = \frac{(P + \lambda \sigma E H_o)\delta^2}{\mu_o N^2_{Ha}} \left[1 - \frac{1}{\cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha}} \right] \quad \text{for } y \leq y_o \quad (20)$$

Actual experiments on the flow of conducting fluids refer to channels of circular or rectangular cross section. Flow through the latter may be expected to approximate the flow between parallel planes if one side of the rectangle is large compared with the other, the magnetic field being perpendicular to the long side. The electric field in the channel, set up because electric currents cannot flow across its walls, is nonuniform. To represent it as closely as possible (6) the uniform field E in Equations (17) through (20) is adjusted to make the total electric current

$$\sigma \int (E - \lambda v H_o)\, dy \quad (21)$$

flowing between the plates vanish. This gives

$$(P + \lambda \sigma E H_o) = \frac{N_{Ha} P \cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha}}{\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right) \cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N_{Ha}} \quad (22)$$

Substitution of Equation (22) into Equations (17) through (20) gives the following final expressions:

$$v = \frac{P\delta^2}{\mu_o} \frac{\cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha} - \cosh\left(1 - \frac{y}{y_o}\right)\frac{y_o}{\delta} N_{Ha}}{N_{Ha}\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right)\cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N^2_{Ha}} \quad \text{for } y \geq y_o \quad (23)$$

and

$$\tau = P\delta \frac{\sinh \frac{y_o}{\delta} N_{Ha} + \sinh\left(\frac{y}{y_o} - 1\right)\frac{y_o}{\delta} N_{Ha}}{\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right)\cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N_{Ha}} \quad (24)$$

and

$$\tau_o = \frac{P\delta \sinh \frac{y_o}{\delta} N_{Ha}}{\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right)\cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N_{Ha}} \quad (25)$$

and

$$v_o = \frac{P\delta^2}{\delta} \frac{\cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha} - 1}{N_{Ha}\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right)\cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N^2_{Ha}} \quad (26)$$

In the absence of a magnetic field, $N_{Ha} = 0$, Equations (23) through (26) reduce to

$$v = \frac{P\delta^2}{2\mu_o}\left\{1 - \left(\frac{y}{\delta}\right)\right\} - \frac{P\delta y_o}{\mu_o}\left(1 - \frac{y}{\delta}\right) \text{for } y \geq y_o \quad (27)$$

$$\tau = P y \quad (28)$$

$$\tau_o = P y_o \quad (29)$$

$$v_o = \frac{P\delta^2}{2\mu_o}\left(1 - \frac{y_o}{\delta}\right)^2 \text{for } y \leq y_o \quad (30)$$

For Newtonian fluids $\tau_o = 0$, then $y_o = 0$, and the results obtained in Equations (23) through (26) simplify to

$$(P + \lambda \sigma E H_o) = \frac{P N_{Ha}}{\tanh N_{Ha}} \quad (31)$$

$$v = \frac{P\delta^2}{\mu_o} \frac{\cosh N_{Ha} - \cosh \frac{y}{\delta} N_{Ha}}{N_{Ha} \sinh N_{Ha}} \quad (32)$$

$$\tau = P\delta \frac{\sinh \frac{y}{\delta} N_{Ha}}{\sinh N_{Ha}} \quad (33)$$

When $\tau_o = 0$ and $N_{Ha} = 0$, Equations (27) through (33) reduce to the well-known expressions derived in text books.

In Figure 1 dimensionless velocity profiles for $N_{Ha} = 3$, $y_o/\delta = 0.3$ (a Bingham fluid in a magnetic field), $N_{Ha} = 0$, $y_o/\delta = 0.3$ (a Bingham fluid with no magnetic field), $N_{Ha} = 3$, $y_o/\delta = 0$, (Newtonian fluid in a magnetic field), and finally for $N_{Ha} = 0$, $y_o/\delta = 0$, corresponding to the flow of a Newtonian fluid in the absence of a magnetic field, are presented. It should be noted that the value of $\tau_o/P\delta$, corresponding to $y_o/\delta = 0.3$, is obtained to be 0.0746 from Equation (25). Velocity profiles similar to those presented could easily be obtained from the equations presented above for different values of N_{Ha} and y_o/δ or $\tau_o/P\delta$.

The two other physical quantities of practical importance associated with the flow are the flow rate and the average velocity. Integration over the extent of the fluid in the z-direction and over the width 2δ then leads to

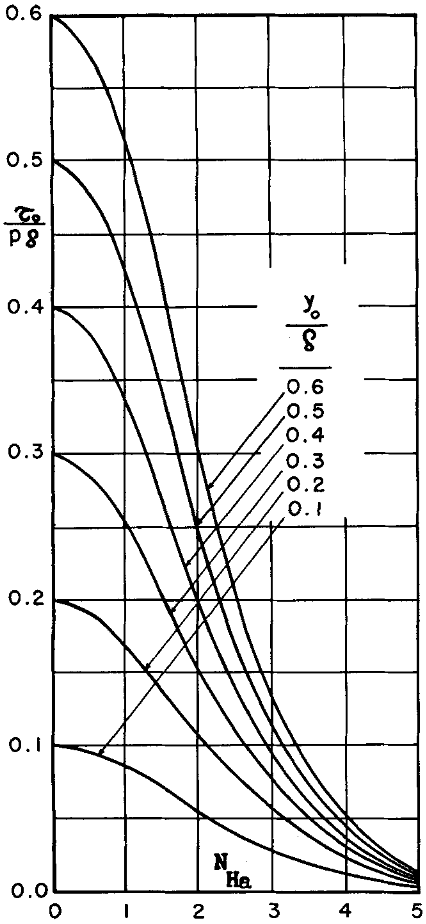

Fig. 4. Variation of y_o/δ, as a function of N_{Ha} and dimensionless shear stress.

the relation among all the basic variables involved:

$$Q = 2\eta \int_0^\delta v \, dy \qquad (34)$$

which is found to be

$$Q_1 = \frac{2 P \delta^3 \eta}{\mu_o} \frac{N_{Ha} \cosh\left(1 - \frac{y_o}{\delta}\right) N_{Ha} - \sinh\left(1 - \frac{y_o}{\delta}\right) N_{Ha} - \frac{y_o}{\delta} N_{Ha}}{N^2_{Ha}\left(\sinh N_{Ha} - \sinh \frac{y_o}{\delta} N_{Ha}\right) \cosh \frac{y_o}{\delta} N_{Ha} + \frac{y_o}{\delta} N^3_{Ha}} \qquad (35)$$

The average velocity is determined from $Q_1/2\delta\eta$.

In the limit as N_{Ha} approaches zero, the rate of flow or the Equation (35) reduces to

$$Q_2 = \frac{2 P \delta^3 \eta}{3 \mu_o}\left\{1 - \frac{3}{2}\frac{y_o}{\delta} + \frac{1}{2}\left(\frac{y_o}{\delta}\right)^3\right\} \qquad (36)$$

and as both N_{Ha} and y_o/δ approach zero, the rate of flow reduces to

$$Q_3 = \frac{2 P \delta^3 \eta}{3 \mu_o} \qquad (37)$$

In order to determine the flow rate of a Bingham fluid in the presence of a magnetic field one can prepare two types of plots. One is the ratio of the flow rate of Bingham fluid in the absence of magnetic field to that with the magnetic field, Q_2/Q_1, (Figure 2), and the other is the ratio of the flow rate of a Newtonian fluid in the absence of a magnetic field to the flow rate of a Bingham fluid in a magnetic field, Q_3/Q_1, (Figure 3). In both cases the pressure gradient is assumed to be identical.

Finally another plot is prepared with Equation (25) to establish a quick passage between $\tau_o/P\delta$ and y_o/δ and N_{Ha} as shown in Figure 4.

OSTWALD AND de WAELE TYPE OF NON-NEWTONIAN FLUID FLOW IN A MAGNETIC FIELD

Substituting Equation (2) into Equation (16) one obtains

$$P + \lambda \sigma H_o (E - \lambda v H_o) + m n \frac{d^2 v}{dy^2}\left(\frac{dv}{dy}\right)^{n-1} = 0 \qquad (38)$$

with

$$a = P + \lambda \sigma H_o E$$

and

$$b = \lambda^2 \sigma H_o^2$$

Equation (38) may be written as

$$\frac{b}{a} v - 1 = \frac{m n}{a} \frac{d^2 v}{dy^2}\left(\frac{dv}{dy}\right)^{n-1} \qquad (39)$$

Substituting

$$\frac{b}{a} v - 1 = u, \frac{du}{d(y/\delta)} = \xi,$$

$$\frac{d^2 u}{d(y/\delta)^2} = \xi \frac{d\xi}{du}$$

one has

$$u = \frac{m n}{b^n} \frac{a^{n-1}}{\delta^{n+1}} \frac{d^2 u}{d(y/\delta)^2}\left\{\frac{du}{d(y/\delta)}\right\}^{n-1} \qquad (40)$$

or

$$u = \frac{m n}{b^n} \frac{a^{n-1}}{\xi^{n+1}} \xi^n \frac{d\xi}{du} \qquad (41)$$

Integrating once one obtains

$$\frac{1}{\delta} \frac{du}{d(y/\delta)} = (u^2 - C_1^2)^{\frac{1}{n+1}} \left\{\frac{b^n (n+1)}{2 m n a^{n-1}}\right\}^{\frac{1}{n+1}} \qquad (42)$$

which gives finally

$$\int \frac{du}{(u^2 - C_1^2)^{\frac{1}{n+1}}} = \left\{\frac{b^n (n+1) \delta^{n+1}}{2 m n a^{n-1}}\right\}^{\frac{1}{n+1}} \frac{y}{\delta} + C_2 \qquad (43)$$

In no other case except for the special cases $n = 0$, and $n = 1$, can the indefinite integral in Equation (43) be expressed by means of a finite number of elementary functional symbols. Therefore it becomes necessary to use the series solution:

With $(n-1)/(n+1) = 2N$, and $u = C_1 \cosh t$ one has

$$\int \frac{du}{(u^2 - C_1^2)^{\frac{1}{n+1}}} = C_1^{2N} \int \sinh^{2N} t \, dt \qquad (44)$$

Performing the integration in Equation (44) with the help of series expansion of the hyperbolic function appearing under the integral sign one obtains

$$C_1^{2N}\left\{(-1)^N \binom{2N}{N} \frac{t}{2^{2N}} + \frac{1}{2^{2N-1}} \sum_{k=0}^{N-1} (-1)^k \binom{2N}{K} \frac{\sinh(2N-2k)t}{(2N-2k)}\right\} = \left\{\frac{b^n (n+1) \delta^{n+1}}{2 m n a^{n-1}}\right\}^{\frac{1}{n+1}} \frac{y}{\delta} + C_2 \qquad (45)$$

With the boundary conditions

$$y = \pm \delta, v = 0,$$

$$u = -1, \cosh t = -\frac{1}{C_1}$$

$$\frac{dv}{dy} = 0, \frac{du}{d(y/\delta)} = 0 \text{ for } y = 0,$$

$$\cosh t = 1, t = 0$$

one finds that $C_2 = 0$, and C_1 has to be determined from the following relationship:

$$C_o^{2N}\left\{(-1)^N \binom{2N}{N} \frac{\cosh^{-1}\left(\frac{1}{C_o}\right)}{2^{2N}} + \frac{1}{2^{2N-1}} \sum_{k=0}^{N-1} (-1)^k \binom{2N}{K} \frac{\sinh(2N-2k)\cosh^{-1}\frac{1}{C_o}}{(2N-2k)}\right\} = \left\{\frac{b^n (n+1) \delta^{n+1}}{2 m n a^{n-1}}\right\}^{\frac{1}{n+1}} \qquad (46)$$

in which C_o is substituted for $(-C_1)$ for convenience in writing. The dimensionless quantity N_{gH}

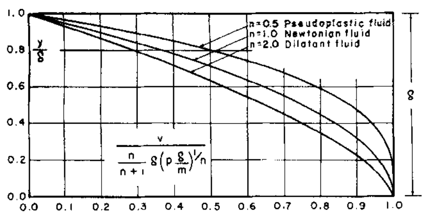

Fig. 5. Velocity profiles for the flow of Ostwald and deWaele type of non-Newtonian fluids.

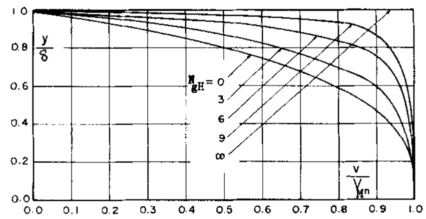

Fig. 7. Pseudoplastic fluid in a magnetic field ($n = 0.5$).

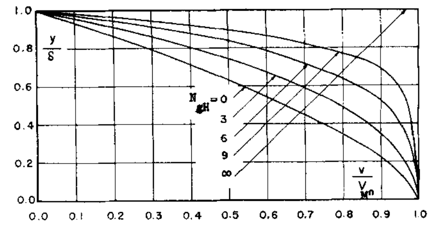

Fig. 8. Dilatant fluid in a magnetic field ($n = 2$).

$$N_{gH} = \left\{ \frac{b^n \delta^{n+1}}{m\, a^{n-1}} \right\}^{\frac{1}{n+1}} = \left\{ \frac{\lambda^2 \sigma H_o^2 \delta^2}{m} \right\}^{\frac{1+2N}{2}} \cdot \left\{ \frac{m}{(P + \lambda \sigma H_o E)\delta} \right\}^{2N} \quad (47)$$

may be regarded as a generalized Hartmann number. For $n = 1$ or for $N = 0$ it reduces to N_{Ha}. As a matter of fact for this special case Equations (45) and (46) reduce to:

$$t = N_{Ha} \frac{y}{\delta} \text{ and } \frac{1}{C_o} = \cosh N_{Ha}$$

which in turn reduces to:

$$v = \frac{P + \lambda \sigma E H_o}{\lambda^2 \sigma H_o^2} \left\{ 1 - \frac{\cosh \frac{y}{\delta} N_{Ha}}{\cosh N_{Ha}} \right\} \quad (48)$$

Equation (48) is identical with the expression which would be obtained by writing $y_o = 0$ in Equation (17). The second special case is obtained by writing $H_o = 0$, which corresponds to the flow of an Ostwald and de Waele type of non-Newtonian fluid between two parallel plates in the absence of a magnetic field. The resulting velocity distribution is given by

$$v = \frac{n}{n+1} \left(\frac{P\delta}{m} \right)^{\frac{1}{n}} \delta \left\{ 1 - \frac{y}{\delta} \right\}^{\frac{n+1}{n}} \quad (49)$$

Velocity profiles corresponding to Equation (49) are shown in Figure 5.

In order to demonstrate the effect of the magnetic field on both the Newtonian and Ostwald and de Waele type of non-Newtonian fluids several figures are presented:

Writing $y = 0$ in Equation (32) one has

$$V_{M1} = \frac{P\delta^2}{\mu_o} \frac{\cosh N_{Ha} - 1}{N_{Ha} \sinh N_{Ha}} \quad (50)$$

Combining Equations (32) and (50) one gets

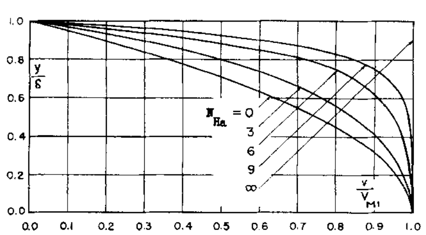

Fig. 6. Newtonian fluid in a magnetic field ($n = 1.0$).

$$\frac{v}{V_{M1}} = \frac{\cosh N_{Ha} - \cosh \frac{y}{\delta} N_{Ha}}{\cosh N_{Ha} - 1} \quad (51)$$

which is plotted in Figure 6 for various values of N_{Ha}. From this figure it is obvious that as the intensity of the magnetic field increases, the distribution of velocity becomes increasingly more uniform. It should be noted that as N_{Ha} increases, the actual value of V_{M1} in Equation (50) decreases rapidly.

Velocity profiles for a representative pseudoplastic fluid ($n = 0.5$) and for a dilatant fluid ($n = 2$) flowing in the presence of a magnetic field are shown in Figures 7 and 8, respectively. The values of v/V_{Mn} as a function of y/δ are computed from Equations (45) and (46) for various values of the generalized Hartmann number N_{gH}. Once again it is evident that the effect of the magnetic field is to make the distribution of velocity more nearly uniform. Consequently the effect of the apparent viscosity is confined to a region near the boundary.

The flow rate for Ostwald and de Waele type of substances flowing in a magnetic field cannot unfortunately be obtained in a closed form as in the case of Bingham fluids. This conclusion follows from a brief inspection of the expressions (45) and (46). A numerical evaluation of the flow rates however is perfectly feasible. Exact solutions for problems of this type exist only for special cases. These cases may or may not correspond to a specific case of practical interest. However the theoretical model often serves as a good approximation to the physical model of interest and predicts qualitative fluid behavior.

NOTATION

a, b = parameters
C = various constants
E = electric intensity
F = body force vector
H = magnetic field strength
J = electric current density
k = dimensionless number
m, m' = rheological constants
n, n' = rheological constants
N = dimensionless number
N_{Ha} = Hartmann number, $(\lambda \delta H_o \sqrt{\sigma/\mu_o})$
N_{gH} = generalized Hartmann number, see Equation (47)
N_{mR} = generalized magnetic Reynolds number, $(\sigma\, m\, V^n_{av}/\delta^{n-2})$
p = pressure intensity
P = pressure gradient, $(-dp/dx)$
Q = various flow rates
t = time, later used as a variable
u = another variable
v = velocity vector
V_{av} = average velocity
V_{M1} = maximum velocity of a Newtonian fluid in a magnetic field
V_{Mn} = maximum velocity of a power-law model fluid in a magnetic field
x, y, z = coordinate axes
y_o = distance to where $|\tau| = \tau_o$

Greek Letters

δ = half of the spacing between the plates
ϵ = dielectric constant
ξ = another variable
η = thickness of the flow along the z-axis
κ = excess electric charge
λ = magnetic permeability
μ_o = viscosity, $(n = 1)$
ρ = density
σ = electrical conductivity
τ = shear stress tensor
τ = shear stress and a rheological constant
Ω = external body forces per unit volume

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Manuscript received September 19, 1960; revision received January 9, 1960; paper accepted January 11, 1960. Paper presented at A.I.Ch.E. Tulsa meeting.